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STOCHASTIC VERSION OF THE ERDÖS–RÉNYI LIMIT THEOREM

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Abstract. We generalize the Erdös–Rényi limit theorem on the maximum of partial sums of random variables to the case when the number of terms in these sums in randomly distributed. Relations between this limit theorem and the spectral theory of random graphs and random matrices are discussed.

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1. INTRODUCTION

The Erdös–Rényi limit theorem concerns the asymptotic behaviour of the random variables

$$\eta(n, k) = \max_{i=1,\ldots,n-k} S_i(k)/k, \qquad S_i(k) = \xi_i + \xi_{i+1} + \ldots + \xi_{i+k}, \tag{1.1}$$

where $\Xi = \{\xi_i\}_{i=1}^{\infty}$ is a family of independent identically distributed (i.i.d.) random variables determined on the same probability space Ω and having zero mathematical expectation $E\xi = 0$. It is assumed that the function

$$\phi(\tau) = Ee^{\xi\tau} \tag{1.2}$$

is determined for $\tau \in I_\xi$, where $I_\xi \subseteq \boldsymbol{R}_+ = (0, +\infty)$.

In [7] it is proved that given $1 < C < \infty$ there exists, with probability 1, a non-random limit

$$\lim_{n\to\infty} \eta(n, [C\log n]) = \alpha \tag{1.3}$$

determined by the relation

$$\inf_{\tau\in I} \phi(\tau)\, e^{-\alpha\tau} = e^{-1/C}. \tag{1.4}$$

In the particular case when ξ_i are given as Bernoulli random variables

$$\xi_i = \zeta_i = \begin{cases} 1 & \text{with probability } 1/2, \\ -1 & \text{with probability } 1/2, \end{cases} \tag{1.5}$$

the convergence (1.3) holds with $C = c\log_e 2$, where $\alpha(c)$ is determined by the relation

$$\frac{1}{c} = 1 - h\left(\frac{1+\alpha}{2}\right) \tag{1.6}$$

with

$$h(t) = -t\log_2 t - (1-t)\log_2(1-t), \qquad 0 < t < 1.$$

It is easy to see that in this case α takes values between the mean value $0 = \alpha(+\infty)$ and the maximum $1 = \alpha(1)$ of random variables ζ_i. Obviously, one can also determine the limit $\alpha(c)$ for the values $c \in (0, 1)$; in this case it is equal to 1.

Further studies give more details about the convergence (1.3); in particular, the convergence in probability was proved and the estimates with probability 1 were derived for the difference

$$\frac{k}{\log k}[\eta(n, k) - \alpha(C)]$$

(see [5]). The Erdös–Rényi theorem has found several applications (see, e.g., [3], [11]) and its various generalizations have been considered (random variables indexed by sets, non-independent identically distributed random variables, random variables in Banach spaces and others).

One more version of this limit theorem is motivated by the studies of spectra of random matrices [8]. Namely, when regarding the weighted adjacency matrix of a random graph, one observes that the spectral norm of such a matrix is bounded from below by the maximum of the sums $S_i(k)$ (1.1), where the number of terms k is distributed at random [8]. Then, in the limit of large dimension of such a sparse random matrix, one faces the problem that can be called the *stochastic version of the Erdös–Rényi limit theorem.* It is clear that in this direction one can find different generalizations of the Erdös–Rényi theorem. In the present paper we give the proof of the results announced previously [9] in the form maximally close to (1.1). We discuss other related settings at the end of the paper.

Let us complete this introduction with expressions of gratitude to Professors A. Rouault and E. Rio for the interest to this work and valuable discussions.

2. MAIN RESULT AND DISCUSSION

Let us consider the family of i.i.d. random variables $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ determined on the same probability space as Ξ, also independent of the family Ξ. These λ_i take values in N according to the law $\Pr\{\lambda = k\} = q(k)$ such that $E\lambda = p$. We assume that the function $\psi_p(t) = Ee^{\lambda t}$, $t \in I_\lambda \subseteq \boldsymbol{R}_+$ exists,

$$\psi_p(t) = \exp\big(p\chi(t)(1+o(1))\big) \quad \text{as } p \to \infty \tag{2.1}$$

and $\chi(t)$ is analytic and satisfies the conditions $\chi(t) \geqslant 0$ and $\chi(0) = 0$. It is easy to deduce from (2.1) that

$$\Pr\{\lambda \geqslant l\} \leqslant \inf_{t\in I_\lambda} \psi_p(t)e^{-tl} = \exp\big(-pf(l/p)(1+o(1))\big) \quad \text{as } p \to \infty, \tag{2.2}$$

where

$$f(y) = \sup_{t\in I_\lambda}[yt - \chi(t)]. \tag{2.3}$$

We assume that $f(y)$ is the steep function, i.e. it takes value $+\infty$ when y goes beyond the domain of the definition of f. Then by the Gärtner–Ellis theorem (see e.g. [6])

$$\Pr\{\lambda \geqslant yp\} = \exp\big(-pf(y)(1+o(1))\big), \ y \geqslant 0, \quad \text{as } p \to \infty. \tag{2.4}$$

Let us note that $f(y)$ is a non-negative strictly convex monotone function. It attains its minimal value at the point $y' = (E\lambda)/p = 1$.

THEOREM 2.1. *Let us consider the sums*

$$S_i(\lambda_i) = \xi_i + \xi_{i+1} + \ldots + \xi_{i+\lambda_i}, \tag{2.5}$$

where $\{\xi_i\}$ *are as in* (1.1), *and determine*

$$\tilde{\eta}(n, p) = \max_{i=1,\ldots,n} \eta_i(n, p), \qquad \eta_i(n, p) = S_i(\lambda_i)/p.$$

There exists with probability 1 *a non-random limit*

$$\lim_{n\to\infty} \tilde{\eta}(n, C\log n) = \tilde{\alpha} \tag{2.6}$$

determined by the following relations:

(i) *If*

$$D(\tilde{\alpha}/y) = \max_{\tau\in I_\xi}[(\tilde{\alpha}\tau)/y - \log\phi(\tau)], \tag{2.7}$$

then $\tilde{\alpha} = \tilde{\alpha}(C)$ *is determined by the relation*

$$\inf_{y \geq 0} [f(y) + yD(\tilde{\alpha}/y)] = 1/C \tag{2.8}$$

that generalizes (1.4).

(ii) *In the case of Bernoulli random variables* $\xi_i = \zeta_i$ (1.5), *the convergence* (2.6) *holds with* $\tilde{\alpha} = \tilde{\alpha}(c)$ *determined by the relation* (cf. (1.6))

$$\inf_{y \in (\tilde{\alpha}, +\infty)} \left\{ f(y) + y\left[1 - h\left(\frac{1}{2} + \frac{\tilde{\alpha}}{2y}\right)\right]\right\} = \frac{1}{c}, \tag{2.9}$$

where $c = C/\log_e 2$.

To compare this theorem with results of [7], let us consider first the case (ii) of Bernoulli random variables. The next simplifying assumption is that λ_i have the Poisson distribution with parameter p. This makes (2.5) close to the model arising in the studies of sparse random matrices (see the end of this paper). One can easily derive that in this case $\chi(t) = e^t - 1$, $I_\lambda = \boldsymbol{R}_+$, and

$$f(y) = \begin{cases} 0 & \text{if } y \in (0, 1), \\ y(\log y - 1) + 1 & \text{if } y \in [1, \infty). \end{cases}$$

The function

$$g_a(y) = y\left[1 - h\left(\frac{1}{2} + \frac{a}{2y}\right)\right] \tag{2.10}$$

is positive and strictly decaying on $(a, +\infty)$; the maximum is attained at a and $g_a(a) = a$, $g_a'(a) = -\infty$. The solution of (2.9) exists for all $c \in (0, +\infty)$ and $\lim_{c \to \infty} \tilde{\alpha}(c) = 0$. This coincides with the value of $\alpha(+\infty) = 0$ given by (1.6).

It is not hard to show that $\tilde{\alpha}(c) > \alpha(c)$ for all finite values of c. Moreover, (2.9) implies that $\tilde{\alpha}(c)$ increases infinitely as $c \to 0$. This means that

$$\lim_{1 \ll p \ll \log_2 n} \tilde{\eta}(n, p) = +\infty, \tag{2.11}$$

while the corresponding value of $\alpha(c)$, $c \to 0$, remains equal to 1. This is an important difference between the usual and stochastic cases of the Erdös–Rényi limit theorem (see Section 4).

The reason for (2.11) is that in the limit $c \to 0$ the averaging in (2.5) is not sufficient and $\tilde{\eta}(n, p)$ really searches for the maximum of variables η_i. This is provided by those variables that have almost all ζ_i equal to 1; since one can see large deviations of the number of terms in $S_i(\lambda_i)$ with respect to p, one can obtain infinite values of $\tilde{\eta}(n, p)$ in (2.11).

Thus, we conclude that the large fluctuations of $q(l)$ in the scale p are responsible for (2.11). This proposition is supported by the following observation. Let us forget the Poisson distribution of λ and assume that there exists a finite interval $Y \subset (0, \infty)$ such that $q([py]) = o(e^{-p})$ for all $y \in \bar{Y} = \boldsymbol{R} \backslash Y$. Then we determine $f(y)$ as $+\infty$ on $\bar{Y}$ and still consider (2.9). In this case $\sup_c \tilde{\alpha}(c)$ is finite. Finally, we observe that if $f(y)$ is close to the Dirac δ-function $\delta(y-1)$, then $\tilde{\alpha}(c)$ is close to the values $\alpha(c)$ given by (1.5).

Summing up these arguments, we arrive at the conclusion that $\lim_{c \downarrow 0} \tilde{\alpha}(c) = \infty$ provided the fluctuations of λ_i around p are sufficiently large.

It is the general case of finite but unbounded random variables ξ_i, the limit $\tilde{\alpha}(C)$ as $C \to 1$ can be infinite already in the classical case of $\lambda_i \equiv k = [C \log n]$.

3. PROOF OF THEOREM 2.1

As in [7], we give the proof of the item (ii) concerning the Bernoulli random variables ζ_i, and then we describe the changes needed to prove Theorem 2.1 in the general case.

Let us show that for any positive ε

$$\Pr\{\tilde{\eta}(n, c\log_2 n) \geqslant \tilde{\alpha}+\varepsilon\} = O(n^{-\delta}) \quad \text{as } n \to \infty, \tag{3.1}$$

where $\delta > 0$ depends only on ε. We start with the elementary inequality

$$\Pr\{\sup_{i=1,\ldots,n} \eta_i(n, p) \geqslant x\} \leqslant \sum_{i=1}^{n} \Pr\{\eta_i(n, p) \geqslant x\} = n \Pr\{\eta_1(n, p) \geqslant x\},$$

where we used the fact that η_i are identically distributed. Observing that $\{\omega\colon \eta_1 \geqslant x\} \subset \{\omega\colon \lambda \geqslant px\}$, we can write

$$\Pr\{\eta_1(n, p) \geqslant x\} = \sum_{l \geqslant px} q(l) \Pr\{S(l) \geqslant px\}. \tag{3.2}$$

Using the Stirling formula, we obtain

$$\frac{1}{2^l} \sum_{(l+px)/2 \leqslant j \leqslant l} \binom{l}{j} = \sum_{(l+px)/2 \leqslant j \leqslant l} \frac{2^{-j\log_2(j/l)-(l-j)\log_2(1-j/l)}}{\sqrt{2\pi j(1-j/l)-1}}(1+o(1)).$$

Elementary computation shows that the last sum is estimated by its first term multiplied by a constant. Then we obtain the inequality

$$\Pr\{S(l) \geqslant px\} \leqslant \frac{U}{\sqrt{l}} 2^{-l[1-h(1/2+(xp)/(2l))]}. \tag{3.3}$$

If $x \geqslant \tilde{\alpha}+\varepsilon$, then there exists $\delta > 0$ such that

$$f(y)+y\left[1-h\left(\frac{1}{2}+\frac{x}{2y}\right)\right] = f(y)+g_x(y) > \frac{1+\delta}{c} \tag{3.4}$$

for all $y \geqslant 0$. It is clear that the minimal value of $f(y)$ is $f(1) = 0$. Since $f(y)$ is strictly convex and monotone, it is continuous. Let us denote by z the value such that $f(y) \leqslant \delta/(2c)$ for $1 \leqslant y \leqslant z$. Then for all $0 \leqslant y \leqslant z$

$$cg_x(y) > 1 + \delta/2. \tag{3.5}$$

Using monotonicity of $g_x(\cdot)$, we derive from (3.3) the inequality

$$\Pr\{\eta \geqslant x\} \leqslant \Big(\sum_{px \leqslant l \leqslant pz} + \sum_{pz \leqslant l}\Big) \frac{U}{\sqrt{l}} q(l)\, 2^{l[h(1/2+(xp)/(2l))-1]}$$

$$\leqslant U \sum_{xp \leqslant l \leqslant pz} 2^{-pg_x(l/p)} + 2^{-pg_x(z)(1+o(1))} \sum_{l \geqslant pz} q(l).$$

Taking into account that $p = c\log_2 n$, using (3.5) and combination of (2.2) and (3.4), we obtain

$$\Pr\{\eta_1(n, p) \geqslant x\} = O(n^{-1-\delta/4})$$

because the number of terms in the first sum is of the order $O(\log_2 n)$. The relation (3.1) is proved.

To prove the almost sure estimate, we follow the scheme of [7]. Let us consider the sequence of random variables

$$\tilde{\eta}_j \equiv \tilde{\eta}(e^{(j+1)/C} - 1, j).$$

Then (3.1) implies the convergence of the series $\sum_j \Pr\{\tilde{\eta}_j > \tilde{\alpha}\}$. Now, taking into account that $\tilde{\eta}(n, C\log n) \leqslant \tilde{\eta}_j$ for all n such that $e^{j/C} \leqslant n \leqslant e^{(j+1)/C} - 1$, we obtain the relation

$$\Pr\{\limsup_{n\to\infty} \tilde{\eta}(n, C\log n) \leqslant \tilde{\alpha}\} = 1.$$

This completes the estimate from above of the limit $\lim \tilde{\eta}(n, c\log_2 n)$ for the case of Bernoulli random variables.

In the general case one can use the inequality (see e.g. [1])

$$\Pr\{S(l) \geqslant px\} = (2\pi l b_l)^{-1/2}\, e^{-lD(x/y)}, \tag{3.6}$$

where $0 < b \leqslant b_l \leqslant B < \infty$, instead of (3.3). The remaining part of the proof repeats the arguments presented above.

Now let us show that $\Pr\{\max_i \eta_i < \tilde{\alpha} - \varepsilon'\}$ vanishes as $n \to \infty$. To do this, we take an integer m and determine the subsets of Ω

$$B_n(m) = \{\omega \in \Omega\colon \sup_{i=1,\dots,n} \lambda_i < m\}.$$

The next observation is that the events

$$A_k(n, m) = \{\omega\colon \eta_{km+1}(n, p) \leqslant x \,|\, B_n(m)\}$$

are jointly independent for all $0 \leqslant k \leqslant n(m)-1$, $n(m) = [n/m]$. Thus, we can write

$$\Pr\{\sup_i \eta_i \leqslant x \,|B_n(m)\} = \prod_{k=1}^{n(m)} \Pr\{A_k(n, m)\} \tag{3.7}$$

$$= \left(\frac{\Pr\{\eta_1 \leqslant x \cap B_n(m)\}}{\Pr(B_n(m))}\right)^{n(m)},$$

where we have put $\eta_1 = \eta_1(n, p)$. Using the elementary relations

$$\Pr\{F \cap B_n(m)\} \leqslant 1 - \Pr\{\bar{F} \cap B_n(m)\}$$

and

$$\Pr\{D \cap B_n(m)\} = \Pr\{D\} - \Pr\{D \cap \overline{B_n(m)}\} \geqslant \Pr\{D\} - \Pr\{\overline{B_n(m)}\}$$

with $F = \{\omega\colon \eta_1 \leqslant x\}$ and $D = \bar{F}$, we can write

$$\Pr\{\eta_1 \leqslant x \cap B_n(m)\} \leqslant 1 - \Pr\{\eta_1 > x\} + \Pr\{\overline{B_n(m)}\}. \tag{3.8}$$

Let us consider $\Pr\{\eta_1 > x\}$. If $x < \tilde{\alpha}(c) - \varepsilon$, then there exist $\delta' > 0$ and $z' > 1$ such that

$$f(y) + y\left[1 - h\left(\frac{1}{2} + \frac{x}{2y}\right)\right] \geqslant \frac{1-\delta'}{c} \quad \text{for all } y \geqslant z_1. \tag{3.9}$$

The Stirling formula implies the following inequality inverse to (3.3):

$$\frac{1}{2^l} \sum_{(l+px)/2 \leqslant j \leqslant l} \binom{l}{j} \geqslant \frac{u}{\sqrt{l}} 2^{-l[1-h(1/2+(xp)/(2l))]}. \tag{3.10}$$

Using this estimate and remembering about the monotonicity of the function $g_x(\cdot)$ in (2.10), we derive from (3.2) the relation

$$\Pr\{\eta_1 > x\} \geqslant n^{-cz_1[1-h(1/2+x/(2z_1))]} \sum_{y \geqslant pz_1} q(l).$$

Now (2.4) together with (3.9) imply that

$$\Pr\{\eta_1 > x\} = O(n^{-1+\delta'}). \tag{3.11}$$

In the general case, one can use (3.6) instead of (3.10) and obtain (3.11).

Let us estimate $\Pr\{\overline{B_n(m)}\} \leqslant n \Pr\{\lambda_i \geqslant m\}$. We use again (2.2) and observe that if z'' is such that $f(z'') \geqslant 3/C$, then

$$\Pr\{\overline{B_n(m)}\} = O(n^{-2}),\ m = pz'', \quad \text{as } n \to \infty. \tag{3.12}$$

Now we can derive from (3.7), (3.8), (3.11) and (3.12) that

$$\Pr\{\sup_i \eta_i \leqslant x \,|B_n(pz')\} \leqslant \left(\frac{1 - O(n^{-1+\delta'})}{1 - O(n^{-2})}\right)^{n/(pz'')} = O(\exp(-n^{\delta'/2})).$$

Finally, writing the inequality

$$\Pr\{\sup_i \eta_i \leqslant x\} \leqslant \Pr\{\sup_i \eta_i \leqslant x \,|\, B_n(m)\} \Pr(B_n(m)) + \Pr\{\overline{B_n(m)}\}$$

with $m = pz''$, we get

$$\Pr\{\tilde{\eta}(n, p) < \tilde{\alpha} - \varepsilon'\} = O(n^{-2}).$$

Therefore $\Pr\{\liminf_{n\to\infty} \tilde{\eta}(n, C\log n) \geqslant \tilde{\alpha}\} = 1$. This completes the proof of Theorem 2.1. ∎

4. APPLICATIONS TO RANDOM GRAPHS AND RANDOM MATRICES

Let us consider the adjacency matrix A of a simple graph Γ with the sets of vertices and edges denoted by V and E, respectively. If $|V| = N$ and the vertices are enumerated, then A is an $N \times N$ real symmetric matrix with the entries

$$A_{ij}^{(N)} = \begin{cases} 1 & \text{if the edge } e(i, j) \in E, \\ 0 & \text{if } e(i, j) \notin E, \end{cases} \quad i, j = 1, \ldots, N. \tag{4.1}$$

The set of eigenvalues of $A^{(N)}$ is often called the *spectrum* of Γ (see [3]).

One of the models of random graphs (see e.g. [2]) is determined by the ensemble $\{A^{(N,p)}\}$ of matrices whose entries $\{a_{ij}, i \leqslant j\}$ are given as a family of jointly independent random variables with distribution

$$a_{ij} = \begin{cases} 1 & \text{with probability } p/N, \\ 0 & \text{with probability } 1 - p/N. \end{cases}$$

Having a random graph $\Gamma^{(N,p)}$, one can ask about the asymptotic behaviour of its spectrum when $N \to \infty$, in particular, what happens with the maximal (minimal) eigenvalue of $A^{(N,p)}$. This question was addressed in [8] in a more general setting than (4.1).

Namely, the random matrix ensemble $W_{ij}^{(N,p)} = a_{ij} w_{ij}$ has been studied, where $\{w_{ij}, i \leqslant j\}$ are jointly independent random variables, independent also of $\{a_{ij}\}$. It is assumed that the probability distribution of w_{ij} has all odd moments zero $m_{2k+1} = 0$ and $m_{2k} \leqslant k^{(1+\tau)k}$ with $\gamma \geqslant 0$, $k \geqslant 1$. Under these conditions, it was shown that the spectral norm of the matrix

$$\hat{W}^{(N,p)} = \frac{1}{\sqrt{p}} W^{(N,p)}$$

in the limit $N, p \to \infty$, converges with probability 1 to the limits

$$\|\hat{W}^{(N,p)}\| = \begin{cases} 2v & \text{if } p = O((\log n)^{1+\gamma}), \\ +\infty & \text{if } p = O((\log n)^{1-\gamma}) \end{cases} \tag{4.2}$$

for any $\gamma > \tau$. Here we have put $v = \sqrt{Ew_{ij}^2}$, $i, j = 1, \ldots, N$.

Modifying slightly computations of [8], one can show that the same convergence (4.2) is valid for the spectral norm of $\|p^{-1/2} A^{(N,p)}\|$ with $\gamma > 0$.

To study the limit of $p = C \log N$, one has to carry out more accurate analysis than that of [8]. One of the possible results can be obtained by using Theorem 2.1. Indeed, one can write the inequality

$$\|\hat{W}^{(N,p)}\|^2 \geqslant \max_{i=1,\dots,n} \|\hat{W}^{(N,p)} e(i)\|^2 \equiv \max_{i=1,\dots,n} T_i(N, p),$$

where $e(i)_j = \delta_{ij}$. Observing that

$$T_i(N, p) = \frac{1}{p} \sum_{j=1}^{N} a_{ij} w_{ij}^2 \geqslant \frac{1}{p} \sum_{j \geqslant i}^{N} a_{ij} w_{ij}^2 \equiv \hat{T}_i(N, p),$$

one faces the same problem as described in Theorem 2.1. Indeed, $p\hat{T}_i(n, p)$ is given by the sum of independent random variables and the number of terms is given by $\hat{\lambda}_i = \sum_{j=i}^{N} a_{ij}$ that approaches the Poisson random variables λ_i with parameters $ip/N \leqslant p$, respectively. Thus $p\hat{T}_i(n, p)$ resembles $S_i(\lambda_i)$ in (2.5) with ξ replaced by $\hat{\xi}_j = w_{ij}^2$. Let us put

$$H(N, p) = \sup_{i=1,\dots,N} T_i(N, p) \quad \text{and} \quad \hat{H}(N, p) = \sup_{i=1,\dots,N} \hat{T}_i(N, p). \tag{4.3}$$

Consequently, the first difference between (2.5) and (4.3) is the following:

$$E\hat{\xi}_j = v^2 > 0. \tag{4.4}$$

However, it is easy to check that Theorem 2.1 remains valid in the case of (4.4). The relations (2.6)–(2.8) do not change provided ϕ in (1.2) is replaced by $\hat{\phi}(\tau) = Ee^{\tau\hat{\xi}}$. In this case $\hat{\alpha}(+\infty) = v^2$.

The following proposition is true:

$$\lim_{N \to \infty} H(N, C \log N) \leqslant \hat{\alpha}(C), \tag{4.5}$$

where $\hat{\alpha}(C)$ is determined by (2.7) and (2.8) in terms of $\hat{\phi}(\tau)$. We put inequality in (4.5) because the parameters of random variables $\hat{\lambda}_i$ are of the order p provided $i \sim 1$ but decrease to zero when i increases up to N. This is another difference between $\hat{H}(N, p)$ and $\tilde{\eta}(n, p)$ determined in (2.5).

In this connection, it would be interesting to develop an analogue of the Erdös–Rényi limit theorem for maxima of $\hat{T}_i$ and of T_i. It is natural to expect that $\lim H(N, C \log N) = \hat{\alpha}(C)$. Of special interest is the study of asymptotic behaviour of $\|\hat{W}^{(N, C\log N)}\|^2$ also because in the limit $C \to \infty$ it is four times greater than that of $\hat{\alpha}(C)$.

Using the adjacency matrix $A^{(N,p)}$, it is shown in [10] that its maximal eigenvalue is closely related with the maximal degree Δ of a random graph. Since the asymptotic behaviour of Δ is fairly well studied, this gives an important source of information on the spectra of random graphs. It could be interesting to find the limit of the spectral norm of $A^{(N,p)}$ in dependence on C, where $p = C\log N$, $N \to \infty$.

The behaviour of sums of type (2.5) is interesting by itself in the following aspect. Assume that the random variables λ_i are such that $E\lambda_i = p$ but the second moment of λ_i does not exist. Then it is interesting to know whether the border $p \sim \log n$ still remains to be the critical one for the maxima $\tilde{\eta}(n, p)$.

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